Groups, periodic planes and hyperbolic buildings.

ALINA VDOVINA

Mathematisches Institut
Beringstr. 1, Bonn 53115
e-mail: alina@math.uni-bonn.de

Abstract

We give an elementary construction of polyhedra whose links are connected bipartite graphs, which are not necessarily isomorphic pairwise. We show, that the fundamental groups of some of our polyhedra contain surface groups. In particular, we construct polyhedra whose links are generalized m-gons. The polyhedra of this type are interesting because of their universal coverings, which are two-dimensional hyperbolic buildings with different links. The presentation of the results is done in the language of combinatorial group theory.

Introduction

We will call a *polyhedron* a two-dimensional complex which is obtained from several oriented p-gons by identification of corresponding sides. Consider a point of the polyhedron and take a sphere of a small radius at this point. The intersection of the sphere with the polyhedron is a graph, which is called the *link* at this point.

We consider the polyhedra such that all links of all vertices are connected bipartite graphs.

We will say, that a polyhedron P is an (m,n)-polyhedron, if the girth of any link of P is at least m and each face of P is a polygon with at least n edges.

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Let P be a (m, n)-polyhedron such that m and n satisfy the inequality $mn \geq 2(m + n)$, which appear in small cancellation theory [13]. The minimal solutions of the equality are $(6, 3), (4, 4), (3, 6)$.

The universal covering of a (m, n)-polyhedron with the metric introduced in [1],p.165 is a complete metric space of non-positive curvature in the sense of Alexandrov and Busemann, [11].

In chapter 2 we construct a family of (m, n)-polyhedra with a given even number of sides of every face, such that the links of vertices are bipartite graphs, in general, nonisomorphic.

Definition. We say that sets $\mathcal{G}_1$, $\mathcal{G}_2$,..., $\mathcal{G}_k$ of connected bipartite graphs are *compatible*, if the number of white vertices of all graphs of every set is equal to the same number n and there are bijections between sets of white vertices for every pair $\mathcal{G}_1$, $\mathcal{G}_j$, $j = 2, ..., k$, preserving the degrees of vertices.

Theorem 1. Let $\mathcal{G}_1$, $\mathcal{G}_2$,..., $\mathcal{G}_k$ be compatible sets of connected bipartite graphs and $k \geq 1$. Then there exists a family of finite polyhedra with $2k$-gonal faces and links at vertices isomorphic to the graphs from $\mathcal{G}_1$, $\mathcal{G}_2$,..., $\mathcal{G}_k$.

Let X be a universal covering of a (m, n)-polyhedron P. Then, by a result of Gromov [12], p.119, the fundamental group G of P is hyperbolic iff X does not contain a flat Euclidean plane. A natural question is the following: whether G contains $\mathbb{Z} \oplus \mathbb{Z}$ if X contains a flat plane.

This question is a particular case of more general problem, formulated by Ballmann and Brin for polygonal (m, n)-complexes [1],p.166. They solved it for all $(3, 6)$-complexes [1] and for $(6, 3)$-complexes in the case of euclidean buildings [2].

In chapter 3 we show that fundamental group of our polyhedron contains $\mathbb{Z} \oplus \mathbb{Z}$ if its universal covering contains a flat plane.

In chapter 4 we apply the construction from Chapter 2 to prove existence of periodic planes in some hyperbolic buildings.

Definition. A *generalized m-gon* is a graph which is a union of subgraphs, called apartments, such that:

1. Every apartment is a cycle composed of $2m$ edges for some fixed m.

2. For any two edges there is an apartment containing both of them.

3. If two apartments have a common edge, then there is an isomorphism between them fixing their intersection pointwise.

Definition. Let $\mathcal{P}(p, m)$ be a tessellation of the hyperbolic plane by regular polygons with p sides, with angles π/m in each vertex where m is an integer. A *hyperbolic building* is a polygonal complex X, which can be expressed as the union of subcomplexes called apartments such that:

1. Every apartment is isomorphic to $\mathcal{P}(p, m)$.

2. For any two polygons of X, there is an apartment containing both of them.

3. For any two apartments $A_1, A_2 \in X$ containing the same polygon, there exists an isomorphism $A_1 \rightarrow A_2$ fixing $A_1 \cap A_2$.

Let C_p be a polyhedron whose faces are p-gons and whose links are generalized m-gons with $mp > 2m + p$. We equip every face of C_p with the hyperbolic metric such that all sides of the polygons are geodesics and all angles are π/m. Then the universal covering of such polyhedron is a hyperbolic building, see [9].

In the case $p = 3$, $m = 3$, i.e. C_p is a simplex, we can give a euclidean metric to every face. In this metric all sides of the triangles are geodesics. The universal coverings of these polyhedra with the euclidean metric are euclidean buildings, see [2], [3]. The metric characterization of euclidean buildings can be found in [10].

It is known (cf. [14]), that a *generalized m-gon* is a connected, bipartite graph of diameter m and girth $2m$, in which each vertex lies on at least two edges. A graph is *bipartite* if its set of vertices can be partitioned into two disjoint subsets such that no two vertices in the same subset lie on a common edge. The vertices of the one subset we will call black vertices and the vertices of the other subset the white ones. The *diameter* is the maximum distance between two vertices and the *girth* is the length of a shortest circuit.

Let G be a connected bipartite graph on $q + r$ vertices, q black vertices and r white ones. Let $\mathcal{A}$ and $\mathcal{B}$ be two alphabets on q and r letters respectively, $\mathcal{A} = \{x_1, x_2, \ldots, x_q\}$ and $\mathcal{B} = \{y_1, y_2, \ldots, y_r\}$. We mark every black vertex with an element from $\mathcal{A}$ and every white vertex with an element from $\mathcal{B}$.

We will also define an incidence tableau of such a graph in the following way: the first element of each line is a white vertex and

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all other elements are black vertices incident to this white vertex. Different lines correspond to different white vertices.

Example. For the smallest complete bipartite graph (generalized 2-gon) with four vertices, two black x_1, x_2 and two white y_1, y_2, the incidence matrix will be

y_1: x_1 x_2
y_2: x_1 x_2

Define a graph G', called the *dual* to G, in the following way: the graph G' can be obtained from G by changing black vertices to white and vice-versa. (It is easy to see that G and G' are isomorphic as ordinary graphs.)

Similarly, we can define the incidence tableau by "inverse" way, as correspondence of black vertices to white ones. For the graph from the example we have

x_1: y_1 y_2
x_2: y_1 y_2

1   Polygonal presentation.

Definition. Suppose we have n disjoint connected bipartite graphs $G_1, G_2, \ldots G_n$. Let P_i and Q_i be the sets of black and white vertices respectively in G_i, $i = 1, ..., n$; let $P = \cup P_i, Q = \cup Q_i$, $P_i \cap P_j = \emptyset$ $Q_i \cap Q_j = \emptyset$ for $i \neq j$ and let λ be a bijection $\lambda : P \to Q$.

A set $\mathcal{K}$ of k-tuples $(x_1, x_2, \ldots, x_k)$, $x_i \in P$, will be called a *polygonal presentation* over P compatible with λ if

(1) $(x_1, x_2, x_3, \ldots, x_k) \in \mathcal{K}$ implies that $(x_2, x_3, \ldots, x_k, x_1) \in \mathcal{K}$;

(2) given $x_1, x_2 \in P$, then $(x_1, x_2, x_3, \ldots, x_k) \in \mathcal{K}$ for some $x_3, \ldots, x_k$ if and only if x_2 and $\lambda(x_1)$ are incident in some G_i;

(3) given $x_1, x_2 \in P$, then $(x_1, x_2, x_3, \ldots, x_k) \in \mathcal{K}$ for at most one $x_3 \in P$.

If there exists such $\mathcal{K}$, we will call λ a *basic bijection*.

Polygonal presentations for $n = 1$, $k = 3$ were listed in [6] with the incidence graph of the finite projective plane of order two or three as the graph G_1. They were called triangle presentations in that paper, because the case $k = 3$ was considered. Polygonal presentations for $k > 3$ were considered in [15].

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2 Construction of polyhedra.

We can associate a polyhedron K on n vertices with each polygonal presentation $\mathcal{K}$ as follows: for every cyclic k-tuple $(x_1, x_2, x_3, \ldots, x_k)$ from the definition we take an oriented k-gon on the boundary of which the word $x_1 x_2 x_3 \ldots x_k$ is written. To obtain the polyhedron we identify the corresponding sides of our polygons, respecting orientation. We will say that the polyhedron K *corresponds* to the polygonal presentation $\mathcal{K}$.

Lemma[15] A polyhedron K which corresponds to a polygonal presentation $\mathcal{K}$ has graphs $G_1, G_2, \ldots, G_n$ as the links.

Remark. Consider a polygonal presentation $\mathcal{K}$. Let s_i be the number of vertices of the graph G_i and t_i be the number of edges of G_i, $i = 1, \ldots, n$. If the polyhedron K corresponds to the polygonal presentation $\mathcal{K}$, then K has n vertices (the number of vertices of K is equal to the number of graphs), $k \sum_{i=1}^{n} s_i$ edges and $\sum_{i=1}^{n} t_i$ faces, all faces are polygons with k sides.

Proof of Theorem 1. By the Lemma, to construct the polyhedron with given links, it is sufficient to construct a corresponding polygonal presentation.

By the definition of compatible sets of bipartite graphs, there are bijections $\alpha_j, j = 2, \ldots, k$, from the set of white vertices of $\mathcal{G}_1$ to the set of white vertices $\mathcal{G}_j$ preserving the degrees of white vertices.

We mark white vertices of $\mathcal{G}_i, i = 1, \ldots, k$, by letters of an alphabet $\mathcal{A}_i, i = 1, \ldots, 2k$, $\mathcal{A}_i = \{x_1^i, \ldots, x_q^i\}$, such that the bijections $\alpha_j, j = 2, \ldots, k$, are induced by the indexes of letters , i.e. $\alpha_j(x_m^1) = x_m^j$, $j = 2, \ldots, k$, We mark black vertices of $\mathcal{G}_i, i = 1, \ldots, k$ by letters of an alphabet $\mathcal{B}_i = \{y_1^i, y_2^i, \ldots, y_r^i\}$. So, every edge of $\mathcal{G}_i, i = 1, \ldots, k$ can be presented in a form $(x_m^i y_l^i)$, $m = 1, \ldots, n, l = 1, \ldots, r$.

Having such a set of bijections $\alpha_j, j = 2, \ldots, k$ of white vertices we can choose bijections $\beta_j, j = 2, \ldots, k$ of the set of edges of $\mathcal{G}_1$ to the set of edges of $\mathcal{G}_j$ which preserves $\alpha_j, j = 2, \ldots, k$. Let $\beta_j(x_m^1 y_l^1) = x_m^j y_{l_j}^j$, then we take the cyclic word $(x_m^1, y_{l_1}^1, \ldots, x_m^j, y_{l_j}^j, \ldots, x_m^k, y_{l_k}^k)$, $m = 1, \ldots, n, l_t = 1, \ldots, r, t = 1, \ldots, k$, to the set $\mathcal{P}$. We will prove, that $\mathcal{P}$ is a polygonal presentation.

For this we need k more families of graphs $\mathcal{H}_i, i = 1, \ldots, k$, such that every graph H is contained in $\mathcal{H}_i$ if and only if it is dual for a graph G from $\mathcal{G}_i, i = 1, \ldots, k$, and every graph from $\mathcal{G}_i, i = 1, \ldots, k$, has its dual in $\mathcal{H}_i, i = 1, \ldots, k$.

We mark black vertices of $\mathcal{H}_i, i = 1, ..., k$ by letters of an alphabet $\mathcal{A}_i, i = 1, \ldots, 2k$, $\mathcal{A}_i = \{x_1^i, \ldots, x_q^i\}$ and we mark white vertices of $\mathcal{H}_i, i = 1, ..., k$ by letters of an alphabet $\mathcal{B}_i = \{y_1^i, y_2^i, \ldots, y_r^i\}$.

The bijection λ from the set P of all black vertices of graphs from $\mathcal{G}_j, j = 1, ..., k$ and $\mathcal{H}_j, j = 1, ..., k$ to the set Q of all white ones is defined by labels: $\lambda(x_m^j) = x_m^j$ and $\lambda(y_l^j) = y_l^i$.

It is necessary to check all axioms of the polygonal presentation for an arbitrary $2k$-tuple $(x_m^1, y_{l_1}^1, ..., x_m^j, y_{l_j}^j, ..., x_m^k, y_{l_k}^k)$, $m = 1, ..., n, l_t = 1, ..., r, t = 1, ..., k$:

(1) In our construction we take all cyclic permutations of each $2k$-tuple.

(2) Let's consider an arbitrary $2k$-tuple from $\mathcal{P}$. There are two possibilities: when the tuple starts with an element from $\mathcal{A}_j$ or $\mathcal{B}_j$, namely $(x_m^j, y_{l_j}^j, ..., x_m^k, y_{l_k}^k, ..., x_m^{j-1}, y_{l_1}^{j-1})$ or

$(y_{l_j}^j, x_m^{j+1}, ..., x_m^k, y_{l_k}^k ... x_m^{j-1}, y_{l_1}^{j-1})$. In the case of

$(x_m^j, y_{l_j}^j, ..., x_m^k, y_{l_k}^k ... x_m^{j-1}, y_{l_1}^{j-1})$ we have, that $\lambda(x_m^j)$ is a white vertex x_m^j of some graph G from $\mathcal{G}_j$ and $y_{l_j}^j$ is a black vertex of G and x_m^j and $y_{l_j}^j$ are incident in G by the construction of $\mathcal{P}$.

In the case of $(y_{l_j}^j, x_m^{j+1}, ..., x_m^k, y_{l_k}^k ... x_m^1, y_{l_1}^1)$ $\lambda(y_{l_j}^j)$ is a white vertex $y_{l_j}^j$ of some graph H from $\mathcal{H}_j$ and x_m^{j+1} is a black vertex of H and x_m^{j+1} and $y_{l_j}^j$ are incident in H by construction of $\mathcal{P}$.

(3) Since $\alpha_j, j = 2, ..., k$ and $\beta_j, j = 2, ..., k$ are bijections, then there is a unique word in $\mathcal{P}$, which contains given subword of length two. This proves the property (3) of the polygonal presentation.

The Theorem is proved.

We will denote polyhedra constructed in the proof of the Theorem 1 P_k.

3 $\mathbb{Z} \oplus \mathbb{Z}$ as a subgroup of the fundamental group of P_2.

In this chapter we give an euclidean metric to every face of P_2. In this metric all sides of the squares are geodesics and angles are $\pi/2$.

Definition. A plane in the universal covering of P_2 is *periodic*, if it is stabilized by a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup of the fundamental group of P_2.

Theorem 2. If the universal covering U of P_2 contains a flat plane F, then U contains a periodic plane and $\pi(P_2)$ contains a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, where $\pi(P_2)$ is the fundamental group of P_2.
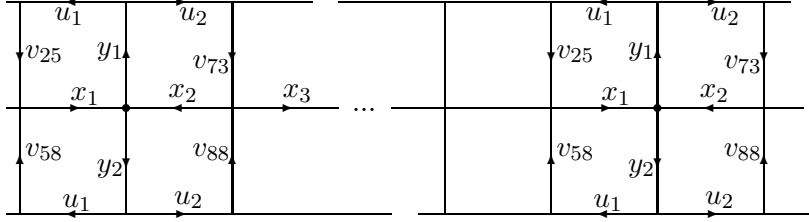

Figure 1.

Proof. All words of the polygonal presentation for P_2 have form $x_i y_j u_i v_l$. So, in the flat plane F there are x, y, u, v-lines. Let's consider an infinite strip S consisting of x-line L and all squares, which contain common edges from L (fig.1). Let v be an internal vertex of S with entering edges x_1, x_2 and leaving edges y_1, y_2. Because of the finiteness of the polyhedron, there is a finite number of x's and y's. So, in our infinite strip S there is another internal vertex w with entering edges x_1, x_2 and leaving edges y_1, y_2. Let the distance between v and w be s(the distance is the number of x-edges). Let's consider the rectangle R which consists of a part of L between v and w and all squares, which contain common edges from L (fig.1). Consider the x-word U of length s $x_2 x_3 ... x_1$ written between v and w. We define the word W to be obtained from U by replacing x by u. Then, on the boundary of R we can read a word $y_2^{-1} y_1 W^{-1} y_1^{-1} y_2 W$. We consider a plane F' in U tesselated by R. This tesselation exists, because the sum of all angles at any vertex of it is equal to 2π (the degree of every vertex is four and each angle is $\pi/2$). The subgroup of $\pi(P_2)$, generated by $a = y_2^{-1} y_1$ and $b = W^{-1}$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ and acts on F' uniformly.

The Theorem 2 is proved.

Let's note, that polyhedra of type P_2 are particular cases of polygonal complexes $(4, 4)$, which were considered in [1].

4 Periodic planes in hyperbolic buildings.

In this chapter we consider polyhedra whose faces are $2k$-gons and whose links are generalized m-gons with $km > m + k$. We equip every face of the polyhedra with the hyperbolic metric such that all sides of the polygons are geodesics and all angles are π/m. Then the universal covering of such polyhedra are hyperbolic buildings, see [9].

In the hyperbolic case the notion of periodicity should be extended to the action of surface groups of genus $g \geq 2$.

Definition. We will say, that a tesselated plane is *periodic*, if there exists a genus $g \geq 1$ sufrace group acting on it uniformly.

It is natural question whether fundamental group of a hyperbolic polyhedron always contains a surface group, if its universal covering contains a hyperbolic plane. In this chapter we will prove that this is true for some hyperbolic buildings. For the proofs we will need some notations concerning Wicks forms.

Definition An *oriented Wicks form* is a cyclic word $w = w_1 w_2 \ldots w_{2l}$ (a cyclic word is the orbit of a linear word under cyclic permutations) in some alphabet $a_1^{\pm 1}, a_2^{\pm 1}, \ldots$ of letters $a_1, a_2, \ldots$ and their inverses $a_1^{-1}, a_2^{-1}, \ldots$ such that

(i) if a_i^ϵ appears in w (for $\epsilon \in \{\pm 1\}$) then $a_i^{-\epsilon}$ appears exactly once in w,

(ii) the word w contains no cyclic factor (subword of cyclically consecutive letters in w) of the form $a_i a_i^{-1}$ or $a_i^{-1} a_i$ (no cancellation),

(iii) if $a_i^\epsilon a_j^\delta$ is a cyclic factor of w then $a_j^{-\delta} a_i^{-\epsilon}$ is not a cyclic factor of w (substitutions of the form $a_i^\epsilon a_j^\delta \longmapsto x$, $a_j^{-\delta} a_i^{-\epsilon} \longmapsto x^{-1}$ are impossible).

An oriented Wicks form $w = w_1 w_2 \ldots$ in an alphabet A is *isomorphic* to $w' = w_1' w_2'$ in an alphabet A' if there exists a bijection $\varphi : A \longrightarrow A'$ with $\varphi(a^{-1}) = \varphi(a)^{-1}$ such that w' and $\varphi(w) = \varphi(w_1)\varphi(w_2)\ldots$ define the same cyclic word.

An oriented Wicks form w is an element of the commutator subgroup when considered as an element in the free group G generated by $a_1, a_2, \ldots$. We define the *algebraic genus* $g_a(w)$ of w as the least positive integer g_a such that w is a product of g_a commutators in G.

The *topological genus* $g_t(w)$ of an oriented Wicks form $w = w_1 \ldots w_{2e-1} w_{2e}$ is defined as the topological genus of the oriented compact connected

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surface obtained by labeling and orienting the edges of a $2e-$gone (which we consider as a subset of the oriented plane) according to w and by identifying the edges in the obvious way.

Proposition *The algebraic and the topological genus of an oriented Wicks form coincide (cf. [7],[8]).*

We define the *genus $g(w)$* of an oriented Wicks form w by $g(w) = g_a(w) = g_t(w)$.

Consider the oriented compact surface S associated to an oriented Wicks form $w = w_1 \ldots w_{2e}$. This surface carries an immerged graph $\Gamma \subset S$ such that $S \setminus \Gamma$ is an open polygon with $2e$ sides (and hence connected and simply connected). Moreover, conditions (ii) and (iii) on Wicks form imply that Γ contains no vertices of degree 1 or 2 (or equivalently that the dual graph of $\Gamma \subset S$ contains no faces which are $1-$gones or $2-$gones). This construction works also in the opposite direction: Given a graph $\Gamma \subset S$ with e edges on an oriented compact connected surface S of genus g such that $S \setminus \Gamma$ is connected and simply connected, we get an oriented Wicks form of genus g and length $2e$ by labeling and orienting the edges of Γ and by cutting S open along the graph Γ. The associated oriented Wicks form is defined as the word which appears in this way on the boundary of the resulting polygon with $2e$ sides. We identify henceforth oriented Wicks forms with the associated immerged graphs $\Gamma \subset S$, speaking of vertices and edges of oriented Wicks form.

The formula for the Euler characteristic

$$\chi(S) = 2 - 2g = v - e + 1$$

(where v denotes the number of vertices and e the number of edges in $\Gamma \subset S$) shows that an oriented Wicks form of genus g has at least length $4g$ (the associated graph has then a unique vertex of degree $4g$ and $2g$ edges) and at most length $6(2g - 1)$ (the associated graph has then $2(2g - 1)$ vertices of degree three and $3(2g - 1)$ edges).

Definition. We will say that word W is obtained from a word U by a non-cancelling substitution Φ, if we substitute every letter of U such that there is no cancellations between $\Phi(y_i)$ and $\Phi(y_j)$ whenever $y_i y_j$ is a subword of U.

Obviously, from a disc with the word U on its boundary we can get a genus g surface by identification of sides (with the same labels

respecting orientation) if and only if a word U is obtained from a Wicks form of genus g by non-cancelling substitution.

Theorem 3. If $I_{2k}, k \geq 3$ is a right-angled hyperbolic building which apartments are hyperbolic planes tesselated by polygons with $2k$ sides, then I_{2k} contains a periodic plane under the action of genus $g = 2k - 4$ surface group.
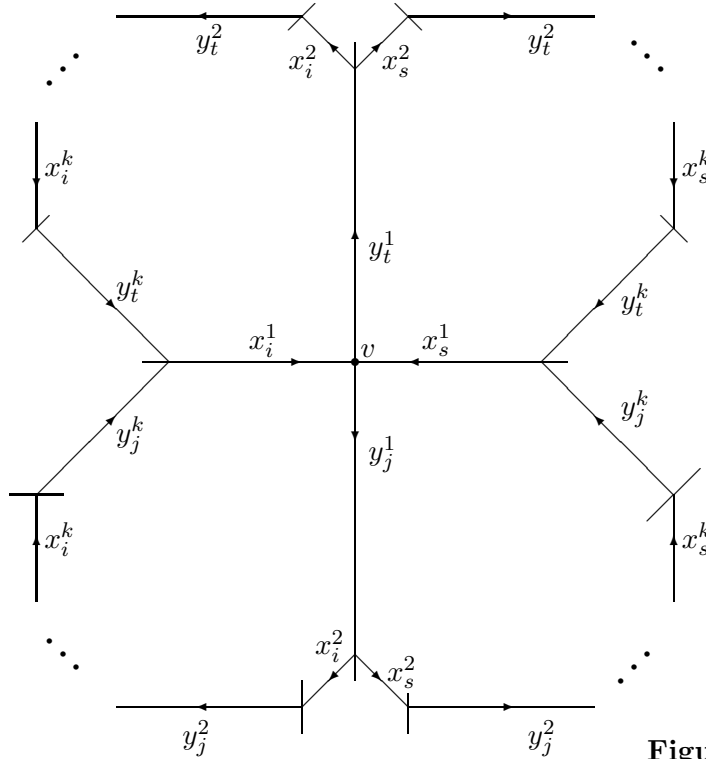

Figure 2.

Proof. Since a right-angled hyperbolic building with given local data is unique ([4] [5]), then it can be obtained as a universal covering of a polyhedron, described in [15]. As G it is necessary to take a complete bipartite graph. This is also a particular case of the construction from the chapter 1, when every set of graphs consists from the same graph G, which is a complete bipartite graph. Let's consider any apartment in $I_{2k}, k \geq 3$. It is a hyperbolic plane tesselated by regular right-angled polygons with $2k$ sides, such that all vertices of the tesselation have degree four. Consider a vertex v such that edges x_i^1 and x_s^1 are entering v and y_t^1 and y_j^1 leaving v (fig.2).There are exactly four polygons containing v:

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$x_i^1 y_j^1 x_i^2 y_j^2 ... x_i^k y_j^k, \ x_i^1 y_t^1 x_i^2 y_t^2 ... x_i^k y_t^k,$
$x_s^1 y_j^1 x_s^2 y_j^2 ... x_s^k y_j^k, \ x_s^1 y_t^1 x_s^2 y_t^2 ... x_s^k y_t^k.$
Consider the word

$$W = x_i^2 y_j^2 ... x_i^k y_j^k (x_i^2 y_t^2 ... x_i^k y_t^k)^{-1} x_s^2 y_t^2 ... x_s^k y_t^k (x_s^2 y_j^2 ... x_s^k y_j^k)^{-1},$$

written on the boundary of the region D, consisting of polygons
$x_i^1 y_j^1 x_i^2 y_j^2 ... x_i^k y_j^k, \ x_i^1 y_t^1 x_i^2 y_t^2 ... x_i^k y_t^k,$
$x_s^1 y_j^1 x_s^2 y_j^2 ... x_s^k y_j^k, \ x_s^1 y_t^1 x_s^2 y_t^2 ... x_s^k y_t^k$ (fig.2). The word W can be obtained from a Wicks form

$$U = a_i^2 b_j^2 ... a_i^k b_j^k (a_i^2 b_t^2 ... a_i^k)^{-1} b_t^2 ... a_s^k (b_j^2 ... a_s^k b_j^k)^{-1},$$

by a non-cancelling substitution, defined as follows: $a_i^n = x_i^n, n = 3, ..., k, a_s^n = x_s^n, n = 3, ..., k, b_i^n = y_i^n, n = 2, ..., k-1, b_t^n = y_t^n, n = 2, ..., k-1, b_j^k = y_j^k (y_t^k)^{-1}, a_i^2 = (x_s^2)^{-1} x_i^2$. The graph of U is on fig.3.
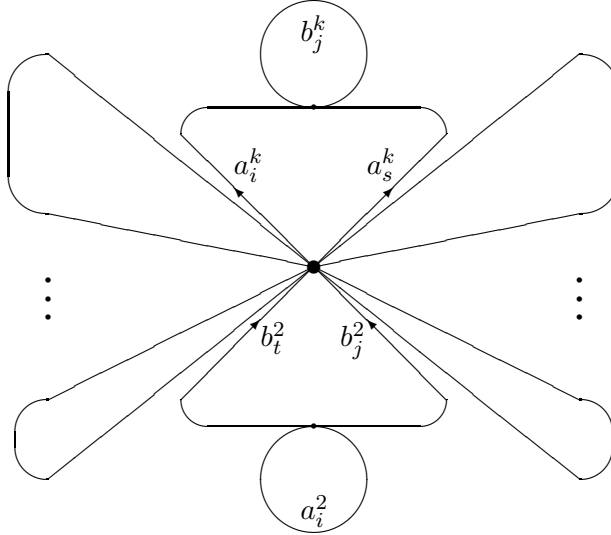

Figure 3.

We can tesselate the hyperbolic plane by the region D, according to the word W, since W is quadratic and the sum of all angles at every vertex of the tesselation is 2π.

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The graph of U has $4k - 6$ edges and 3 vertices. The formula for the Euler characteristic gives $3 - (4k - 6) + 1 = 2 - 2g$, where g is the genus of U. So $g = 2k - 4$.

The genus of the word U and therefore of W is $2k - 4$ and the hyperbolic plane tesselated by D is a periodic plane under the action of genus $g = 2k - 4$ surface group.

The Theorem is proved.

Theorem 4. Let $I_{m,4}$ be a hyperbolic building, obtained from the hyperbolic polyhedron P_4 with a generalized m-gon as a link, $m = 3, 4, 6, 8$, then $I_{m,4}$ contains a periodic plane under the action of genus $g = m - 1$ surface group.
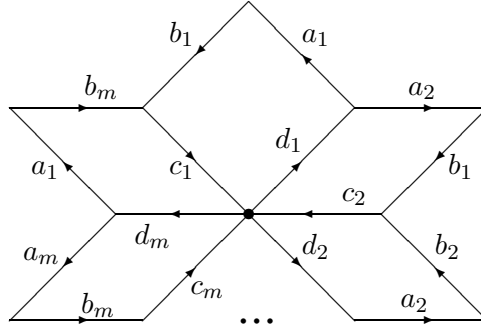

Figure 4.

Proof. Let's consider any apartment in $I_{m,4}$. It is a hyperbolic plane tesselated by polygons with 4 sides, such that all vertices of the tesselation have degree $2m$. Consider a vertex v such that edges $c_1, ..., c_m$ are entering v, $d_1, ..., d_m$ leaving v and entering and leaving edges alternate (fig.4). There are exactly $2m$ polygons containing v: $a_1 b_1 c_1 d_1$, $a_1 b_2 c_1 d_2$, $a_2 b_2 c_2 d_2$,..., $a_m b_m c_m d_m$, $a_m b_1 c_m d_m$. Consider the region D consisting of those polygons and the word

$$W = a_1 b_1 b_2^{-1} a_1^{-1} a_2 ... a_m b_m b_1^{-1} a_m^{-1},$$

written on the boundary of D.

We can tesselate the hyperbolic plane by the region D, according to the word W, since W is quadratic and the sum of all angles at every vertex of the tesselation is 2π.

The genus of the Wicks form W is $m - 1$ and the hyperbolic plane tesselated by D is a periodic plane under the action of genus $g = m - 1$ surface group, the graph of W is on the fig.5.
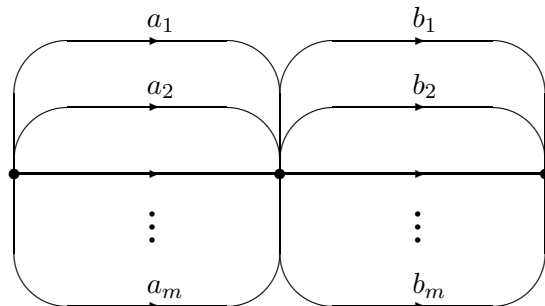
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The Theorem is proved.

Figure 5.

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